Strategic and conceptual challenges experienced by first-year students while attempting to solve problems that require mathematical modelling

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Context

In the midyear holidays in 2010, we presented a two-week academic support course to 74 engineering students and 13 science students in their first year of study, from various universities across South Africa. These students are scholarship holders with the Sasol Inzalo Foundation. The selection of these students is discussed in another paper presented at this forum (Vosloo and Blignaut).

We had two reasons for providing additional support, outside of the university curriculum, to these students. The first reason is that there is a world-wide tendency for greater proportions of the population to become engineers and scientists. This phenomenon is driven mostly by socio-economic needs. The implication is that engineering and science study programs need to take in students who may in the past not have been considered as suitable for such studies, and then need to educate these students to the same high level as before. The second reason, which is related to the first, is that the school system in South Africa is currently struggling to produce sufficient numbers of students who are well prepared for engineering and science courses. As a result, university courses need to adapt, and many engineering and science undergraduate programs in South Africa are already presenting support subjects in addition to 'core' mathematical, science and engineering subjects.

Our design of learning activities for the support course was driven by the concern that students may lack certain conceptual understandings as well as dispositions that are essential in the practice of mathematics and applied mathematics. These concepts or dispositions may be considered as basic as to be school-level, and may therefore not normally be addressed in university courses.

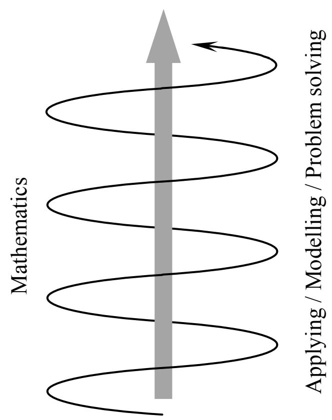
The course was in the first place a remedial action, but it also served a diagnostic purpose. It aimed at addressing conceptual and dispositional shortcomings of school leavers of which we are already acutely aware, based on our experiences over the past six years with FET learners and bridging-year students. At the same time it allowed us to observe students' shortcomings on a more intensive level than before, as the students had already completed their first semester of university studies and could therefore engage with more challenging tasks.

Conceptual framework

The learning activities that we developed can be characterised from three different perspectives. First, it can be characterised in term of the long-term learning path, spanning different courses and learning experiences. Secondly, it can be characterised in terms of the specific mathematical content. And thirdly, it can be characterised in terms of pedagogy.

Long-term learning path

One of the core practices of engineers (and many scientists) is to represent real-world situations in quantitative terms, by means of mathematical models. In the long-term development of a student's mathematical modelling skills and dispositions, there are two parallel developmental paths: learning mathematics, and learning to apply mathematics. The two support each other. One may visualise this as an upwards spiral where the student alternates between learning mathematics, on the one side of the spiral, and learning to apply mathematics, on the other. Students will often revisit the same mathematical content at different stages of their studies, but each time at a higher level.



Different mechanisms of learning – and associated intentions with learning – may occur during this spiral path:

- Learning to solve mathematical (not real-world) problems.
- Learning the mathematics required to solve real-world problems.
- Learning how to solve real-world problems using existing mathematical knowledge.
- Working on real-world problems in order to learn mathematics in the process. This may involve identifying the need for new mathematical learning, and/or making sense of existing mathematical knowledge.

The intention of the learning activities reported here was to develop students' skills and dispositions at solving problems using mathematical modelling, and at the same time to use the problem-solving experience as a means for students to make sense of mathematics that they have already learned.

We choose the problem contexts so as to be easy with regards to the required mathematical procedures, but to be challenging with regards to modelling. Students were confronted with problem situations that were novel to them, where they first needed to make sense of the problem, and to identify the variables, and then had to decide for themselves what mathematical tools to use and how. The real-world situations (contexts) were also chosen so that little prior knowledge from other fields of study (such as physics or mechanics) is required.

The realistic nature of the problem contexts and the project-like way in which we guided students to engage with these problems mimicked the actual work-in-practice of engineers and scientists, thereby allowing students the opportunity for identity development as aspiring engineering and science professionals.

Mathematical content

Our experiences with FET learners indicate that there are certain mathematical concepts of which students have misconceptions or partial understandings that still allow them to solve simple problems, but that will not allow them to solve problems that are more complicated and/or require students to decide for themselves how to approach the problem. We have also observed that such pseudo-conceptions are often barriers to further learning.

The supplementary course that we developed and presented focused largely on co-variation and its mathematical representation. It is reported in the literature that even when students make algebraic expressions to represent relationships in 'word' problems, when such problems involve multiple relationships, they often do not use these expressions to solve the problems. Stacey and MacGregor (2000) observed that high school students (aged 13 to 16) struggled to solve 'word' problems with one unknown, either because they did not make any algebraic representations of the relationships in the problem, or because they made expressions for the different relationships, but did not combine these into a single expression and/or did not use the resulting expression to solve the problem.

Researchers make a distinction between 'algebraic thinking' and 'arithmetic thinking' (Stacy and McGregor, 2000; Thompson, 2007; Gray, Loud and Sokolowski, 2007, 2009). In arithmetic thinking, students think only in terms of known quantities, or quantities that are unknown yet can have only specific values. In algebraic thinking, students use and interpret letter symbols in algebraic expressions to represent quantities that may assume any values. This is also paraphrased as a distinction between operational/procedural thinking (when algebraic expressions are merely used to calculate answers, or to perform operations on), and structural or conceptual thinking (when algebraic expressions can be combined and re-arranged in a way that allows the problem to be solved).

Thompson (2007) points out that "any expression of variation actually involves two quantities" (i.e. covariation). This implies that learners need experiences of engaging with situations that involve co-variation, and in representing such situations in ways that capture both the variables and the relationship between them. We note that awareness of covariation and some proficiency in representing it could be a prerequisite for grasping the idea of the derivative. Whenever two quantities (e.g. time and displacement from a given point) covary, there is a third quantity involved, namely the rate-of-change, which may be constant or may be a variable (Thompson, 2007). The derivative provides a way of algebraically keeping track of a variable rate-of-change.

Gray, Loud and Sokolowski (2009) observed that undergraduate calculus students struggled to reason about variables as generalised numbers and varying quantities when given (single) algebraic expressions containing one or two variable symbols. Of the students they observed, only a third demonstrated algebraic thinking (as defined by them), and there was a positive correlation between the students' algebraic thinking and their performance in calculus.

Bardini, Radford and Sabena (2005) found that grade 11 students who could make algebraic representations of the number of toothpicks in a repeating pattern, later struggled to make algebraic representations of the same pattern when the level of generality of the question was increased (by introducing a parameter with unspecified value). They concluded that students need to learn to act and reason mathematically on a level where algebraic expressions represent "genuine conceptual objects, objects that can only be referred to through signs", and that "students need to cope with the kind of indeterminacy that constitutes a central element of the concepts of variable and parameter."

Trigueros and Ursini (2003) investigated first-year university students' fluency in using letter symbols to represent unknowns, free variables, and variables in situations that involve covariation. They conclude that students have a poor grasp of variables and tend to interpret letter symbols as placeholders for unknowns even if the situation requires a different interpretation. They make the following recommendation:

> "These findings are a call to rethink and analyze the way in which the concept of variable is taught in high school. If we want students to be prepared to work with mathematical concepts at the university level, where new and more complex uses of variable will be introduced and where solution methods rely on a solid understanding of elementary algebra, it is necessary to reconsider the way the concept of variable is taught in elementary algebra. Meanwhile it is necessary to design university courses that can foster students' understanding of the concept of variable."

Malisani and Spagnalo (2008), building on the work of Trigueros and Ursini, found evidence which suggests that the strong establishment of the interpretation of letter symbols as representing unknowns, in the early learning of school algebra, may constrain students from also using letter symbols to represent variables in functional relationships.

To overcome these shortcomings in algebraic thinking, it is suggested that "instruction in courses prior to calculus should include explicit attention to the many different uses of variables" (Gray, Loud and Sokolowski, 2009). Exposing students only to problems that require the setting up of an equation and solving for the unknown, is not sufficient for students to develop algebraic thinking in its full sense.

The problems we used with first-year students, on which we report in this paper, required the students to think algebraically at a higher level than the problems used by Stacey and MacGregor (2000) and Gray, Loud and Sokolowski (2009). Our problems required thinking of multiple variables (unlike the single-unknown problems of Stacey and MacGregor), and were real-life ('word') problems (instead of the purely algebraic problems of Gray, Loud and Sokolowski). In two extended tasks we also required of students to make formulas for the general solutions of problems of a certain kind.

Pedagogy

We worked on the assumption that many students do not develop an understanding of the mathematics they are taught by means of lectures and written explanations alone. They may lack the language, proficiency and reasoning dispositions to absorb and process such explanations. Such students may end up learning mathematics by rote – as much as they can memorise at least – and never become empowered to use mathematics in novel problem contexts.

Some students who do not appropriate concepts in lectures can still develop these concepts by means of experiences of 'figuring things out' (making sense) for themselves. But only a minority of students have the ability and inclination to do this without external motivation and support.

To create guided sense-making experiences, we confronted students with challenging yet accessible real-life problems with which they could engage over extended periods. They had the freedom to try out different approaches, and were encouraged to explain their thinking. We aimed to refrain – as far as possible – from telling students 'what to do', and of rewarding students for correct answers. Instead, we saw our role as that of critical listening and questioning, of introducing counter-examples to show up illogical thinking, and of suggesting investigations that would lead to useful discoveries. The support role we tried to play was much like "scaffolding" as defined by Henningsen and Stein (1997).

Students worked mostly individually, but small group discussions were regularly employed as opportunities for students to explain their thinking (rather than for showing each other 'what to do'). There were no more than 15 students per class, so as to allow for one-on-one interactions between students and the tutor.

International best practice and trends

Our approach to situate the learning of mathematics within real-life problem-solving activities corresponds to the view that situated learning promotes mathematical sense-making, as well as the disposition of looking at reality from a mathematical point of view (Schoenfeld, 1992, 1998, 2007; Gravemeijer, 1999).

Our approach resonates strongly with the ideas of task-based learning and maintaining a high level of challenge, as articulated by Henningsen and Stein (1997).

A recent report of the evolution of mathematics teaching for engineering students, at Australian universities, refers to substantial utilisation of problem-based learning in mathematics courses, and suggests increased use of engineering contexts in the mathematics class (Broadridge and Henderson, 2008).

Description of the innovation

During the first part of the course (2 days) students engaged with two problems that demonstrated the need for making formulas, even if the formulas would not be directly used to calculate answers. During the second part of the course (3 days) students engaged with two situations involving rates-of-change.

Trains activity

The first problem we gave students was a modification of a problem by Sowder et al. (1998):

> There are two parallel railway tracks between the towns Lichtenstein and Schwarenstein. The length of the railway tracks from the one town to the other is 236 kilometres.
>
> On a certain morning, one train needs to travel from Lichtenstein to Schwarenstein and arrive at Schwarenstein not later than 09h40. This train travels at an average speed of 96 km/h.
>
> On the same morning, another train has to travel from Schwarenstein to Lichtenstein and arrive at Lichtenstein not later than 10h00. This train travels at an average speed of 139 km/h.

After initial exploration of the context by the students, and agreeing on the simplifying assumption of constant speed, students were challenged to determine when and where along the railway track the two trains will meet. When they could make no progress with this problem, it was suggested to them to first 'track the progress/positions of both trains with respect to time'.

After students solved the specific problem, mostly arithmetically (i.e. without using variable notation) they were asked to develop a 'tool' to solve the general problem of finding out when any two trains will meet each other, when different speeds for the trains, arrivals times, and distances between the towns could be specified.

Wood factory activity

> A factory produces different type of wood-fibre based composite materials.
>
> For each kg of class A material the following ingredients are needed: 0.4 kg wood fibre, 0.2 kg resin, 0.2 kg filler.
>
> For each kg of class B material the following ingredients are needed: 0.5 kg wood fibre, 0.1 kg resin, 0.2 kg filler.
>
> For each kg of class C material the following ingredients are needed: 0.3 kg wood fibre, 0.3 kg resin, 0.1 kg filler.
>
> The remaining part of the mass of the composites is made up of water that binds with the resin.
>
> a) The factory receives an order for 3500 kg of class A composite, 2300 kg of class B composite and 6100 kg of class C composite. How much of each ingredient is needed to complete this order?
>
> b) At a certain stage the following quantities of the various ingredients are available in the factory stores: 383 kg of wood fibre, 223 kg of resin and 157 kg of filler. The manager decides to use up all these materials by making a batch of class A, class B and class C composite. How much of each kind of composite can be made?

In question a) students were first asked to find out how much of the different materials are needed to make specified amounts of the different products. In question b) they were asked the reversed question of finding out how much of the different products should be made to exactly use up specified amounts of the materials.

Water flow activity

The students observed a practical demonstration of water being siphoned from one container into another, with the flow rate decreasing continuously as the head of water decreases. They were then challenged to produce a good verbal description of how fast the water flows. Only after students reflected and interacted on how fast the water flows and on how this may be accurately described, were they given measurements of water levels (volumes) at different times. They were then guided in a numerical investigation of how the flow rate changes with time, which lead to a piecewise constant description of the flow rate. After more discussions they were provided with a formula for the water level at any point in time, allowing them to calculate effective flow rates over arbitrarily small time intervals. The learning activity continued by guiding students towards a (re)invention of the derivative.

The above approach develops the concept of derivative by employing the goal of describing the covariation between the independent variable and the rate of change of the dependant variable, rather than to introduce the derivate with the simpler purpose of determining the rate of change at a single point. Hence this approach simultaneously develops the concept of covariation.

Volume of a cone activity

The concept of integration was developed using the problem of approximating the volume of a cone. This problem was chosen because it does not require any knowledge from other fields of study (such as physics), because the differential elements can be visualised, because the independent variable is not time (as it is in motion or flow rate problems), and because it does not appear as an integration problem (e.g. where the area under a curve has to be determined). The latter point is very important, because our aim was that students (re)discover the meaning of integration by first working numerically, and attempts by students to integrate (analytically) straight away will bypass this learning objective.

Students were first guided in solving the cone problem numerically. Thereafter the water flow situation was revisited, but this time a formula for the flow rate as a function of time was given, and the question was to find a formula for volume as a function of time. Students first did this numerically, as before for the cone problem.

The water flow situation was used as a demonstration (mostly lectured, but with some student interaction) of how the analytical solution to such a problem can be found, by using the approach of the 'anti-derivative'. It was suggested to students, that if one has an expression for the flow rate, but needs to find a formula for the volume, one may use the following strategy: Try out (guess) a formula for the volume as a function of time, and then differentiate it. If the resultant formula for the derivative is the same as the known formula for the flow-rate, one knows that one has chosen the correct formula for the volume. If not, one modifies one's choice/guess for a formula for volume, and then repeats the process. One continues modifying the chosen/guessed formula for volume until it results in the required derivative (the flow rate formula).

Subsequently, the cone problem was reintroduced, but now with the aim of solving this problem using the 'anti-derivative' approach demonstrated earlier. This time the intention was that students would solve the problem mostly independently. However, the great majority of them could not do that, so in the end the solution was demonstrated in large part.

Analysis of student responses

Trains problem

Students showed a propensity to solve problems in a step-by-step manner, where each step yielded an answer. The answer of one step is then used in the calculation of a following step. Many students used algebraic notation in some calculation steps, in which case they wrote down an equation and then re-arranged it in order to solve for the unknown. Such equations contained only a single variable, and were therefore not expressions of two or more indeterminate quantities varying together (covariation).

The work of typical students to solve the *specific problem* is shown on the following page. Their work is shown in two parts, where the first part concerns only the calculation of departure times, and the second part concerns the calculation of the time when the two trains meet/cross each other. Two variations are shown for the second part of the solution.

The work shown are for valid solution procedures. Many students only arrived at these solutions after some struggling.

Length of trip: Train A	Length of trip: Train B
$96x = 236$	$139x = 236$
$x = \frac{236}{96} \times 60 = 148$ min	$x = \frac{236}{139} \times 60 = 102$ min
Latest departure time: Train A:	Latest departure time: Train B:
arrival time = 9h40min	arrival time = 10h00min
length of trip = 2h28min	length of trip = 1h42min
∴ departure time = 7h12min	∴ departure time = 8h18min

Figure 1: Students' step-by-step approach to solve the trains problem, part one

Note that the calculations above were done step-by-step, each time using the answer of the previous step in the calculation in the next step. Note also the student used the symbol *'x'* in each case to refer to 'the answer', and the same symbol *'x'* was therefore used for different answers.

Train A	Train B
Position from L: ~~$y = 1.6x$~~ $y = 1.6z$	Position from L: $y = 236 - 2.32x$
Position from S: ~~$y = 236 - 1.6x$~~ $y = 236 - 1.6z$	Position from S: $y = 2.32x$

But train A has a head start on train B. $x =$ time travelled by train B $z = x + 66$

According to this information, the new departure times for the trains A and B are 8h18.

… train A only left with 130km to reach town S [after 8h18].

$130 - 1.6x = 2.32x$ $x = \frac{130}{3.92} = 33$min $8h18min + 33min = \mathbf{8h51min}$

Figure 2a: Students' step-by-step approach to solve the trains problem, part two

The student above converted speeds in km/h to speeds in km/min, hence the appearance of 1.6 [km/min] and 2.32 [km/min] in the mathematical expressions.

Note that the student above initially used the same *'x'* to refer to both the 'time travelled by train A' and the 'time travelled by train B', ignoring that the trains have different departure times. She later recognised this omission, and then introduced the symbol *'z'* in addition to *'x'* to differentiate between the two different 'times travelled'.

After that, she could have substituted the expression $z = x + 66$ into $y = 1.6z$, which would have lead her to the simultaneous equations $y = 1.6(x + 66)$ and $y = 236 - 2.32x$. But she did not do this. Instead, she then transformed the problem to make the two departure times the same, by calculating the position of train A (which departed first) at the time when train B departs, and calculating the remaining distance for train A (130km). Thereby she 'got rid' of the problem of different departure times, and no longer needed to substitute one algebraic expression into another.

$130 = v_A t + v_B t$ $\therefore t = \frac{130}{v_A + v_B}$

Figure 2b: Students' step-by-step approach to solve the trains problem, part two, variation

The method shown above again made use of transforming the problem to make the departure times the same.

After students successfully solved a specific problem by following sequence of arithmetical and/or algebraic steps (calculation steps), they struggled to make an algebraic model of the problem, in order to find a *general solution* for arbitrary values of the parameters. Their earlier step-by-step arithmetic solutions were of little help as precursors to making algebraic expressions, because their final calculations steps used intermediate answers from previous calculation steps without showing how these were obtained.

Some students went as far as trying somehow (without justification) to fit all the quantities in the problem statement into a (non-derived) formula that would calculate the time when the two trains meet. They often tried to use the physics formula $s=vt$ to make this final formula, but did so in an uncritical and inappropriate manner. This amounts to trying to take a 'shortcut' to get to the final formula without having to think about the situation.

Some students correctly made the expression $v_A \Delta t_A = d - v_B \Delta t_B$, where v_A and v_B are the speeds of trains A and B, Δt_A and Δt_B are the times travelled up to the point where they meet (relative to the two different departure times), and d is the total distance of railway between the two towns. But then they stopped. They had two unknowns in this equation and could therefore not solve it. They did not consider making the substitutions $\Delta t_A = t - t_{departure_A}$ and $\Delta t_B = t - t_{departure_B}$. We suspect that the reason that they did not think about making such substitutions, is that they never identified the variable 'any point in time' (indicated by *'t'* in the suggested substitutions above). In all of their work prior to the point, they never assigned a symbol to 'any point in time', but only used symbols to refer to quantities that are initially unknown but that can soon be calculated.

When students stopped short of finding a general solution, it was mostly because of the following:

1) Trying to use some known formula in an uncritical and inappropriate manner. *(Misapplication of known formula.)*

2) Guessing how to put all the given quantities into a formula to calculate the answer. *(Guessing formula.)*

3) Not defining unique symbols to represent all the different unknowns. *(Unstructured symbols.)*

4) Not combining algebraic expressions by means of substitution. *(Lack of structural transformation.)*

5) Not identifying variables, i.e. not identifying the covariation between different variables. *(Not identifying variables.)*

The problems listed in 1) and 2) are the result of suspension of sense-making: the students do not justify or test the algebraic representations by reasoning about the situation (mental representation). This lack of connecting between the two modes of representation may be the result of students expecting that in mathematics problems, algebraic expressions are given, and that their task is to calculate answers with the expressions or apply procedures to them (like solving, factorising, or differentiating). In this sense the feedback from one student that "I am now an equation maker, not a user" is encouraging in terms of the effectiveness of the learning activities employed.

The lack of the modelling steps listed in 3) to 5) may be in large part due to students not having experience with problems that cannot be solved by step-wise calculation. The problems currently done at school can typically be solved without the above actions. Moreover, students have experience only with problems that require them to conceive of unknowns (quantities of which the values are soon to be calculated), and not with problems that require them to conceive of variables (quantities that are truly indeterminate). When each expression just contains one unknown, then there is no need to collect more than one indeterminate expression – each representing the covariation between two or more variables – and later combine them to determine all of the variables. As a result students do not (yet) appreciate that it is useful to formulate an expression even if it will not or cannot immediately be used to calculate an answer.

Wood factory problem

The majority of students unknowingly made an invalid assumption that allowed them to solve the reversed problem by means of step-wise calculation. Their approach to solving the problem was to divide each raw material (wood, resin, filler, water) between the different products, according to the proportions given in the problem statement, and then to add up for each product the masses of all the different raw materials allocated to it. This approach is graphically illustrated below. Only two of the four raw materials is shown below, for the sake of brevity, but the same approach was applied to the other two raw materials.. (Note: This is the authors' interpretation of how students reasoned: the representation below is not one made by a student.)

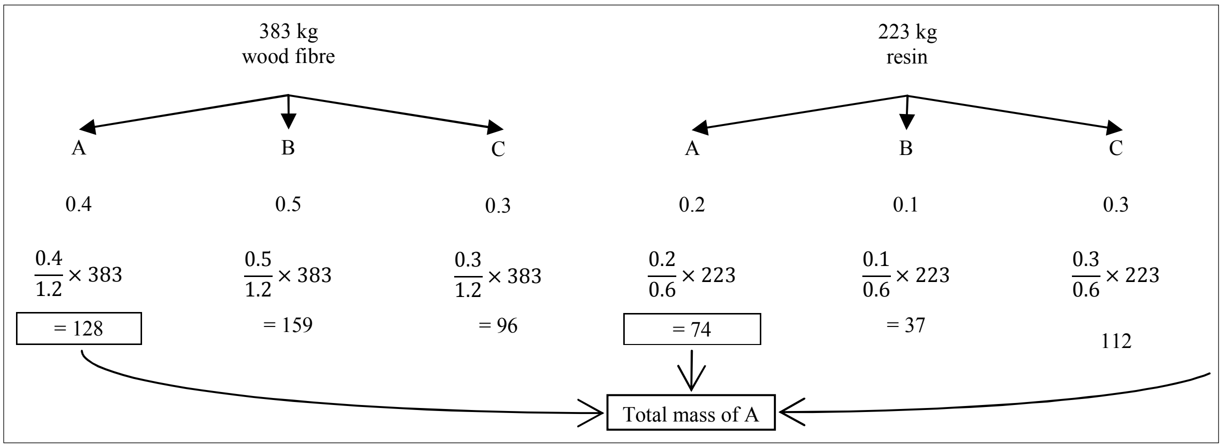


Figure 3: Students' approach to solving the reversed wood factory problem

The problem with the reasoning above is that it assumes that for each 0.4kg of wood fibre being used for product A, 0.5kg of wood fibre will be used for product B, and 0.3kg wood fibre will be used for product C. But that is only true if equal amounts of the different products are made. Since the amounts of the different products are unknown (in fact, the question is to find what these are), it is invalid to assume these amounts to be the same.

The reversed problem can only be solved by first writing formulas for the forwards problem, assigning the given amounts of materials as the results of the respective formulas, and then solving the three equations in three unknowns. One may speculate that students cleverly changed the problem so that it would fit with their expectation of a problem that can be solved by step-wise calculation.

After the lecturer showed up the invalid hidden assumption by means of counter-examples, and explained the forward-reverse structure of the problem by means of a flow diagram, students set out to make the formulas for the forwards problem. But here, once again, many students attempts at making algebraic representations did not make sense. The mechanism by which they lost track was:

6) Literally translating the problem statement into quasi-mathematical language, and in the process using symbols to represent names of objects rather than placeholders for quantities. *(Symbols as names.)*

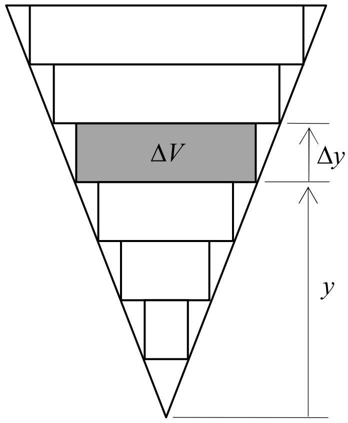
Water flow problem

Most students initially used the formula $Q=V/t$ to calculate flow rate. Once again, they used some known formula (like the one for speed: $v=s/t$) in an uncritical and inappropriate manner. *(Misapplication of known formula.)*

Thereafter, counter-examples were provided to show up that $Q=V/t$ is not useful for comparing how fast the flow was at different times. The counter-examples – as well as an additional question of predicting the volume at a certain time – prompted students to (re-)invent a derived measurement that enables comparison of the how fast the flow was over different time intervals, and enables the making of predictions of volume at different times. That derived measurement is $Q=\Delta V/\Delta t$.

When the siphon activity was previously used with students at the start of a one-year bridging course, they were asked to take experimental measurements right at the start. When the activity was used this time, with students in the middle of their first year of university study, the students were initially given no access to experimental measurements, but we first asked to reason about the flow rate qualitatively. In the former case many students, after having calculated average flow rates over different time intervals, argued that the flow rate over an interval is constant "because they calculated it". In the latter case we did not observe this misconception at all at that point in the learning activity. This raises the suspicion that withholding opportunities for procedural action (doing calculations, in this case) from students promotes mental representation of the situation. But the two groups of students are not quite comparable, so further research is required to investigate this.

Volume of a cone problem



When determining the volume of a cone numerically, many students chose symbols for quantities in way they did not reveal the structure of the problem to them. They were able to solve the problem numerically, but when asked to make an algebraic representation of the problem (which would eventually lead into analytical solution of the integration problem) they struggled. On their drawings of a cone showing a differential element (a short cylinder or disc), they typically used one symbol (e.g. y)for the position of a differential element (as measured from the top or bottom of the cone), and an altogether different symbol (e.g. *d*) for the height of a disc. They did not identify the thickness of a disc as the change in the position from one disc to the next. Thereby the missed the crucial link between the independent variable y, and differential change in this variable Δy. *(Unstructured symbols, not identifying variables.)*

After the presenter pointed out the connection between the independent variable and the differential change in this variable, students made a formula to calculate the differential volume: $\Delta V=f(y)\Delta y$. But when they were then asked to identify the derivative in the situation (so as to be able to apply analytical integration thereafter) they were initially dumbfounded, and then performed one of two nonsensical operations which amount to finding the second derivative. *(Not identifying variables.)*

The one approach used by students was to differentiate the expression for ΔV, in other words to find $\frac{d}{dy}\Delta V$.

The other approach was to mistakenly interpret the formula for ΔV as a formula for V, then to calculate 'V' at two different values of y, and finally to calculate the average rate of change of 'V' with respect to y. This means they calculated $\frac{\Delta V_{y_2}-\Delta V_{y_1}}{y_2-y_1}$.

It was suggested to them at this point first to find an expression for the rate-of-change over an interval (as was done earlier during the water flow activities that aimed to develop the concepts of rate-of-change and derivative). Yet they still did not see the obvious, namely that there is the average rate of change $\Delta V/\Delta y = f(y)$ for an interval in y. There may be different (and inter-related) reasons why the students could not identify the rate-of-change:

- They are not accustomed to think of a rate-of-change in a situation where there is no time dimension.
- They did not at that stage relate their mathematical representation to a mental presentation of the problem, and/or to the previously done numerical solution method, and/or to a double number line representing the covariation between volume (V) and position (y).
- They regarded the expression $\Delta V=f(y)\Delta y$ to be merely a formula for how to calculate the value of the unknown ΔV, and not fully comprehended that this expression represents a relationship between variables.
- They still did not make the connection between rate-of-change over an interval and the derivative.

The point about relating mathematical representations to other kinds of representations reminds us that we often see students (in other contexts) ending a problem with a (nonsensical) mathematical result, but not going back to interpret that result in terms of the problem statement (which would have revealed that lack of sense). We therefore add this to our list of difficulties that students experience in mathematical modelling:

7) Lack of reasoning flexibly – forwards and backwards – with and between different kinds of representations. Revisiting and reinterpreting representation at a later stage, and identifying the relationships between different kinds of representations. *(Flexibly reasoning with and between different kinds of representations.)*

Common themes in students' attempts to solve problems that require mathematical modelling

Underlying to most of the observations discussed above, is the observation that most students were constrained in their ability to think strategically about how to solve novel yet simple problems. In many cases they did not even try to make algebraic expressions for relationships if those expressions could not quickly lead to calculation. Or they made such expressions, but never used them. They seemed to lack the anticipation that gathering various indeterminate relationships in mathematical form, would later enable the expressions to be combined and transformed in a way that would lead to solution of the problem (*cf.* Boero, 2002). In any case, their unstructured and imprecise use of symbols for quantities made it impossible to reason about their mathematical representations in such a manner as to reveal the mathematical structure of the problem.

A secondary, but related, general observation is that students sometimes, once they have made a mathematical representations of a situation, did not 'go back' to reasoning to interpret this relationship.

These two 'foundational' observations may be visualised using the SEM-FORM model of Boero (2002), which illustrates the problem-solving process with a flow diagram that shows transformations between different representations of the problem situation. 'SEM' is defined as mental (internal) representations and non-mathematical external representations (such as drawing and tables). 'FORM' is defined as mathematical external representations.

We modify Boero's model by making distinctions between different kinds of FORM representations: a) representations containing only numbers, b) representations using letter symbols (i.e. algebraic expressions), and c) structured or restructured algebraic representations.

Our distinction between algebraic representation and (re)structured algebraic representation corresponds to the distinction made by Hall et al. (1989), who identified the following processes in algebraic problem-solving:

" Recognising quantitative entities directly contained in or implied by the problem text.
Composing these entities into local relational structures.
Composing relational substructures into larger problem structures.
Recognizing familiar substructural arrangements.
Detecting when constraints are sufficient for solution. "

The figures below use our modification of Boero's SEM-FORM model to contrast the stepwise calculation approach to problem-solving, followed by most students, with an idealised model of making representations and then (re)structuring them before solving the problem.

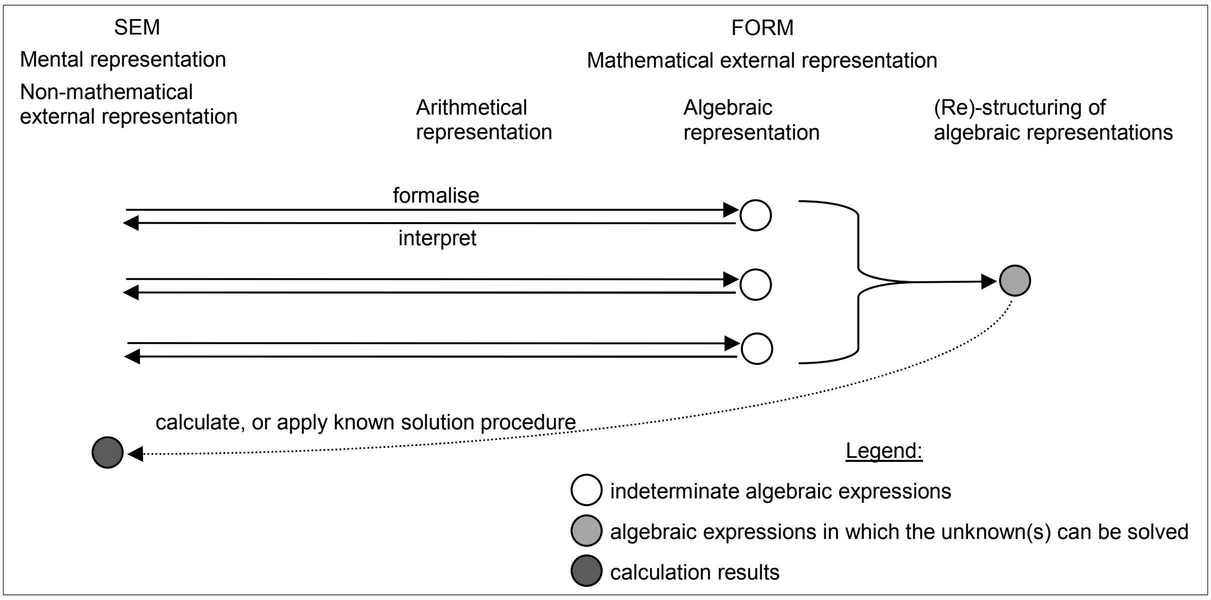


Figure 4a: Transformations between different representations in an idealised algebraic problem-solving approach

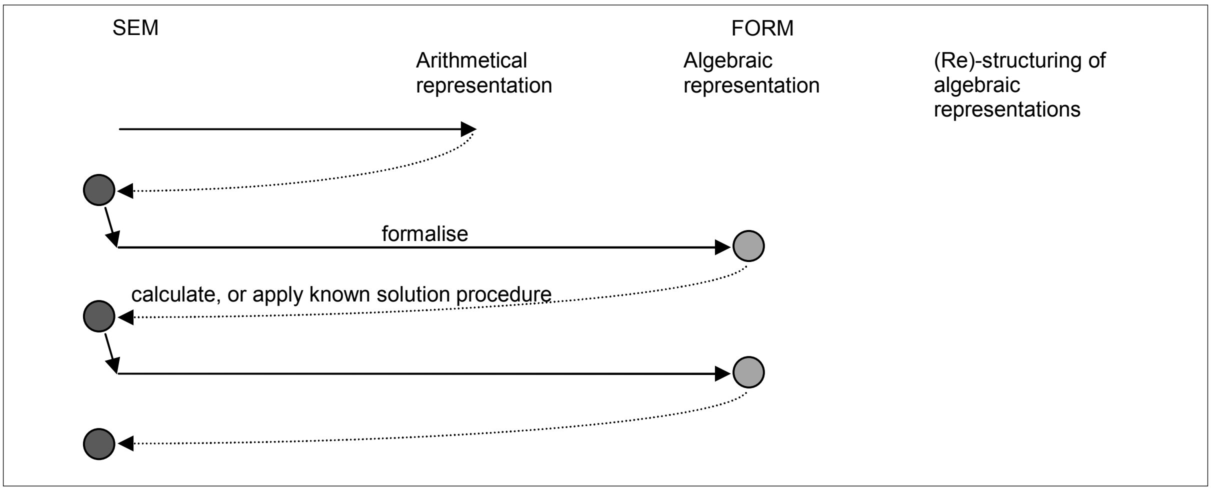


Figure 4b: Transformations between different representations in a typical student's stepwise problem-solving

The stepwise calculation approach is entirely valid, but many of the problems we gave students are not solvable by using this approach. Certain problems can only be solved by moving into the structural part of the algebraic problem-solving process, indicated on the right of the diagrams. Even if a specific problem does not require structural algebraic thinking, finding a general solution to a problem usually does.

The trains problem with specific parameter values that we initially gave students, could be solved by stepwise calculation, and was initially solved in that way by almost all the students we observed. The final step in that process required setting up an equation with one unknown and re-arranging the equation to solve for the unknown. But the general problem for arbitrary values of the parameters required multiple substitutions of one algebraic expression into another (composing functions), and with this most students struggled. In line with the discussion of literature earlier, we do not regard setting up an equation to solve for an unknown as algebraic thinking in its full structural sense, since it only requires thinking about a single unknown, and does not require a global awareness of the all the variables in the problem.

Discussion

We observed that students take mathematical action when they perceive that there is something to be calculated, or when they sense the expectation (of the teacher or question) that some procedure should be performed. Yet they frequently do not take mathematical action when there is nothing that can be readily calculated, and there is no specification or hint of what procedure should be executed.

The 'word' problems that students engage with at school are mostly solvable in one or two arithmetic steps. Only in linear programming do students need to make mathematical expressions to model a situation before they solve the problem. And even in this topic examination questions present students with a sequence of steps to perform – or even provide them with the relevant mathematical expressions – so that students never need the strategic control to decide for themselves what to do.

When faced with a more difficult problem, where they do not initially have a clear plan of how to solve the problem, students stall. They seem to lack the anticipation that certain mathematical actions, although it will not immediately yield calculable results or be part of a pre-identified solution procedure, will collectively lead to emergence of a problem structure and/or to simplification of the problem. Students lack a global awareness of all the variables and relationships in a problem, without which it is impossible to think about the structure of the problem.

When engineering students study applied topics like mechanics, circuit analysis, mass balances, at a later stage in their courses, they will gain experience – if they are successful in studying those topics – with problems that require structural algebraic thinking. Yet at school they have no experience with such thinking, simply because the problems done at school do not require it.

We suggest that it may be beneficial for students to engage with problems that require structural algebraic thinking earlier on in their education, before the above mentioned applied topics are studied, so that their understanding of variables (as opposed to unknowns) and their strategic thinking abilities are already somewhat developed by the time they start with these more advanced topics. Early engagement with such thinking may also help the study of mathematics, for example by making the concept of doing substitutions to simplify integral expressions more accessible.

But some progress towards structural algebraic thinking at an earlier stage may have more than conceptual and technical benefits. Engagement with problems such as we used during the study also provide opportunity for students to change their mathematical dispositions. They may formerly have regarded mathematics action as merely procedural, but may now regard mathematical action to be founded in sense-making and to be purpose-driven.

We suggest that many students may benefit from first-year course content aimed at developing the skills and dispositions of mathematical modelling, employing problems that demand structural algebraic thinking. For example, tutorials could provide students with experiences where they model simple yet novel situations, where they have extended time to explore these situations.

The following continuation of the research presented in this paper is envisaged:

- For a subsequent support course we will design experiments to gather quantitative information to allow us to confirm or reject our qualitative observations.

- The possible benefit of using computer programming as a vehicle for developing structural algebraic thinking will be explored.

- It will be investigated if and how students struggle with applied topics like mechanics, circuit analysis and mass balances, and if such struggles are related to the lack of structural algebraic thinking reported in this paper.

References

Bardini, C., Radford, L., and Sabena, C. (2005). Struggling with variables, parameters, and indeterminate objects or how to go insane in mathematics. In: Chick, H. L. and Vincent, J. L. (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp.129-136. Melbourne: PME.

Boero, P. (2002). Transformation and anticipation as key processes in algebraic problem solving. In: Sutherland, R., Rojano, T., Bell, A., and Lins, R. (Eds.), *Perspectives on School Algebra*, pp. .

Broadridge, P., and Henderson, S. (2008). Mathematics for 21st century engineering students. Final report by the *Australian Mathematical Sciences Institute*. The Carrick Institute for Learning and Teaching in Higher Education, Strawberry Hills, NSW.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, 1 (2), pp. 155–177.

Gray, S. S., Loud, B. J., and Sokolowski, C. P. (2007). Calculus students' difficulties in using variables as changing quantities. *Conference on Research in Undergraduate Mathematics Education,* Tenth Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education, San Diego.

Gray, S. S., Loud, B. J., and Sokolowski, C. P. (2009). Calculus students' use and interpretation of variables: algebraic .s arithmetic thinking. *Canadian Journal of Science, Mathematics and Technology Education,* 9 (2), pp. 59–72.

Hall, R., Kibler, D., Wenger, E., and Tuxaw, C. (1989). Exploring the episodic structure of algebra story problem solving. *Cognition and Instruction,* 6 (3), pp. 223–282.

Henningsen, M., and Stein, M. K. (1997). Mathematical tasks and student cognition: classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education,* 28 (5), pp. 524–549.

Malisani, E., and Spagnalo, F. (2009). From arithmetical thought to algebraic thought: The role of the "variable". *Educational Studies in Mathematics,* 71, pp. 19–41.

Saldanha, L., and Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In: S. B. Berensah and W. N. Coulombe (eds.), *Proceedings of the Annual Meeting of thePsychology of Mathematics Education - North America*. Raleigh, NC: North Carolina State University.

Schoenfeld, A. H. (1992). Learning to think mathematically: problem solving, metacognition, and sensemaking in mathematics. In: D. Grouws (ed.), *Handbook for Research on Mathematics Teaching and Learning*. MacMillan, New York, pp. 334-370.

Schoenfeld, A. H. (1998). Reflections on a course in mathematical problem solving. Research in Collegiate Mathematics Education III, pp. 81–113. In: Schoenfeld, A. H., Kaput, J. and Dubinsky, E. (Eds.). *Research in Collegiate Mathematics Education III*. Providence, American Mathematical Society.

Schoenfeld, A. H. (2007). Problem solving in the United States, 1970–2008: research and theory, practice and politics. *ZDM Mathematics Education*, 39, pp. 537–551.

Sowder, J., Armstrong, B., Lamon, S., Simon, M., Sowder, L., and Thompson, A. (1998) Educating teachers to teach multiplicative structures in the middle grades. *Journal of Mathematics Teacher Education*, 1, pp. 127–155.

Stacey, K., and MacGregor, M. (2000). Learning the algebraic method of solving problems. *Journal of Mathematical Behavior,* 18 (2), pp. 149–167.

Thompson, P. W. (2008). One approach to coherent algebra: Or, it takes 12 years to learn calculus. Draft paper for *Pathways to Algebra Conference*, Mayenne, France. (http://pat-thompson.net/Publications.html)

Trigueros, M, Ursini, S. (2005) First-year undergraduates' difficulties in working with different uses of variable, In: Dubinsky, E., Harel, G., Hitt, F., and Seldon, A. (Eds.). *Research in Collegiate Mathematics Education V*. Providence, American Mathematical Society.